ALMOST SURE LIMIT POINTS FOR DELAYED RANDOM SUMS

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ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive asymmetric stable random variables with a common distribution function F with index $\alpha, 0 < \alpha < 1$. The present work intends to obtain almost sure limit points for a sequence of properly normalized delayed random sums.

KEYWORDS: Law of iterated logarithm, Delayed sums, Delayed random sums, Asymmetric stable law, Almost sure limit points.

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) positive asymmetric stable random variables (r.v.s) with a common distribution function (d.f.) F with index α, $0 < \alpha < 1$. Set $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$ and $T_{a_n} = \sum_{i=n+1}^{n+a_n} X_i = S_{n+a_n} - S_n$, where $\{a_n, n \geq 1\}$ is a non-decreasing sequence of the positive integers of n such that, $0 < a_n \leq n$, for all n and $\frac{a_n}{n} \sim b_n$, where b_n is non-increasing. The sequence $\{T_{a_n}, n \geq 1\}$ is called a (forward) delayed sum sequence [See Lai(1973)].

Let $\{N_n, n \geq 1\}$ be a sequence of positive r.v.s. independent of $\{X_n, n \geq 1\}$ such that $\left| e^{\frac{N_n}{n^\delta}} - 1 \right| \to 0$ almost surely as $n \to \infty$, where $0 < \delta < 1$. Now parallel to the delayed sums T_{a_n}, we introduce delayed random sums as, $M_{N_n} = \sum_{j=n+1}^{n+N_n} X_j = S_{n+N_n} - S_n$.

When X_n's are i.i.d. symmetric stable r.v.s, with index α, $0 < \alpha < 2$ Chover (1966) studied the law of iterated logarithm (LIL) for (S_n), by normalizing in the power. For further developments in Chover's form of LIL see GootyDivanji (2004).

When variance is finite, Lai (1973) had studied the behavior of classical LIL for properly normalized sums (T_{a_n}), at different values of a_n's. For independent, but not identically distributed strictly positive stable r.v.sVasudeva and Divanji (1993) studied the non-trivial limit behavior of delayed sums (T_{a_n}).

In this In this work, we intend to obtain almost sure limit points for (M_{N_n}).

Throughout this Paper, C, ε (small), k(integer), with or without a suffix or super suffix stand for positive constants., whereas a.s. and i.o. mean almost sure and infinitely often respectively. For any sequence (Y_n) of r.v.s $\limsup(\inf) Y_n = \alpha(\beta)$ is to be read as $\limsup Y_n = \alpha$ and $\liminf Y_n = \beta$.

We will frequently use the the following well known results.

2. SOME KNOWN RESULTS

Lemma 1 (Extended Borel-Cantelli Lemma)

Let (E_n) be a sequence of events in a common probability space.

If (i) $\sum_{n=1}^{\infty} P(E_n) = \infty$ and

(ii) $$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{s=1}^{n} P(E_k \cap E_s)}{\left(\sum_{k=1}^{n} P(E_k)\right)^2} \leq C,$$

then $P(E_n \text{ i.o.}) \geq C^{-1}$.

Where C is some positive constant.

For proof, see Spitzer (1964, Lemma p3,p.317)

Lemma 2

Let (A_n) be a sequence of events in a common probability space. If $P(A_n) \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$. Then $P(A_n \text{ i.o.}) = 0$.

For proof, see Nielsen (1961, Lemma 1*,p.385).

Lemma 3

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive asymmetric stable r.v.s with common d.f. F with index $\alpha, 0 < \alpha < 1$. Let $\{N_n, n \geq 1\}$ be a sequence

of positive r.v.s independent of $\{X_n, n \geq 1\}$ such that $\left|e^{\frac{N_n}{n^\delta}} - 1\right| \to 0$ a.s. as $n \to \infty$, where $0 < \delta < 1$. Let $\gamma_n = \left(\log \frac{n}{N_n} + \log\log n\right)$. Then

$$\liminf_{n\to\infty} (\sup) \left(\frac{M_{N_n}}{N_n^{\frac{1}{\alpha}}} \right)^{\frac{1}{\gamma_n}} = 1\,(e^{\frac{1}{\alpha}}) \text{ a.s.}$$

For proof, see Gooty Divanji and K .N. Ravi Prakash (2016 Theorem 1 and Theorem 2).

In the next section, the almost sure limit points for delayed random sums are obtained.

3. ALMOST SURE LIMIT POINTS FOR DELAYED RANDOM SUMS

Theorem 1

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive asymmetric stable r.v.s, with common d.f. F with index α, $0 < \alpha < 1$. Let $\{N_n, n \geq 1\}$ be a sequence of positive r.v.s, independent of $\{X_n, n \geq 1\}$ such that $\left|e^{\frac{N_n}{n^\delta}} - 1\right| \to 0$ a.s. as $n \to \infty$, where $0 < \delta < 1$. Let $\gamma_n = \left(\log \frac{n}{N_n} + \log\log n\right)$. Then all the points in $\left[1, e^{\frac{1}{\alpha}}\right]$ are a.s. limit points of the sequence $\left\{ \left(\frac{M_{N_n}}{N_n^{\frac{1}{\alpha}}} \right)^{\frac{1}{\gamma_n}}, n \geq 1 \right\}$, where, $M_{N_n} = S_{n+N_n} - S_n$.

Proof

Let an arbitrary point in $\left[1, e^{\frac{1}{\alpha}}\right]$ be $e^{\frac{p}{\alpha}}$, $0 \leq p \leq 1$. Observe that for p=0 and p=1, the results follow from Lemma 3. Hence, it is enough to show that for $p \in (0,1)$ there exits sufficiently small $\varepsilon_1 > 0$,

$$P\left(M_{N_n} \geq N_n^{\frac{1}{\alpha}} \left(\frac{n}{N_n} \log n \right)^{\frac{p+\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 0 \qquad (1)$$

and

$$P\left(M_{N_n} \geq N_n^{\frac{1}{\alpha}} \left(\frac{n}{N_n} \log n \right)^{\frac{p-\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 1 \qquad (2)$$

From the condition $\left|e^{\frac{N_n}{n^\delta}} - 1\right| \to 0$ a.s. as $n \to \infty$ implies that there exists some $\varepsilon > 0$ and $\delta \in (0,1)$ such that,

$\left|e^{\frac{N_n}{n^\delta}} - 1\right| < \varepsilon$ implies $u_n < N_n < v_n$ a.s.,(3)

where $u_n = C_1 n^\delta$, $v_n = C_2 n^\delta$, $C_1 = \log(1-\varepsilon)$ and $C_2 = \log(1+\varepsilon)$.

We know that $M_{N_n} = \sum_{j=n+1}^{n+N_n} X_j = S_{n+N_n} - S_n = S_{N_n}$. Since $\{N_n, n \geq 1\}$ be a sequence of positive valued r.v.s. independent of $\{X_n, n \geq 1\}$, by (3) we have

$S_{u_n} < S_{N_n} < S_{v_n}$ a.s. which implies

$$M_{u_n} < M_{N_n} < M_{v_n} \text{ a.s.} \qquad (4)$$

Hence (1) and (2) hold whenever,

$$P\left(M_{v_n} \geq N_n^{\frac{1}{\alpha}} \left(\frac{n}{N_n} \log n \right)^{\frac{p+\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 0, \qquad (5)$$

where $M_{v_n} = S_{n+v_n} - S_n$ and

$$P\left(M_{u_n} \geq N_n^{\frac{1}{\alpha}} \left(\frac{n}{N_n} \log n \right)^{\frac{p-\varepsilon_1}{\alpha}} \text{ i.o.} \right) = 1, \qquad (6)$$

Where $M_{u_n} = S_{n+u_n} - S_n$.

Using (3), one can find some constants $C_3 (> 0)$ and $C_4 (> 0)$ such that,

$$N_n^{\frac{1}{\alpha}} \left(\frac{n}{N_n} \log n \right)^{\frac{p+\varepsilon_1}{\alpha}} \geq C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}}$$

and

$$N_n^{\frac{1}{\alpha}} \left(\frac{n}{N_n} \log n \right)^{\frac{p-\varepsilon_1}{\alpha}} \geq C_4 n^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p-\varepsilon_1}{\alpha}},$$

which in turn (5) and (6) implies that,

$$P\left(M_{v_n} \ge C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}} \text{ i.o.}\right) = 0 \tag{7}$$

and

$$P\left(M_{u_n} \ge C_4 n^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p-\varepsilon_1}{\alpha}} \text{ i.o.}\right) = 1. \tag{8}$$

Let $A_n = \left\{M_{v_n} \ge C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}}\right\}$

and $A_{n+1} = \left\{M_{v_{n+1}} \ge C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}\right\}$

or

$$A^c_{n+1} = \left\{M_{v_{n+1}} \le C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}\right\}$$

As X_n's are i.i.d. positive asymmetric stable r.v.s., we have,

$$P(X \ge x_n) \sim O(x^{-\alpha}). \tag{9}$$

In view of (9), we can observe that, condition (2) of Heyde (1967) is satisfied by $x_n = C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}}$. Following proof of Lemma C in Vasudeva and Divanji (1991), we can show that, $\frac{M_{v_n}}{x_n} \xrightarrow{p} 0$ as $n \to \infty$.

Also since $1 \le \limsup_{n\to\infty} \frac{x_{v_n}}{x_n} \le (1+\varepsilon)^{\frac{1}{\alpha}}$, for some $\varepsilon > 0$, Hyede's Theorem (1967), we can find $C_5 (> 0)$ such that,

$$P(A_n) \le C_5\, n\, P(X_1 \ge x_n). \tag{10}$$

Using (9), there exits some constant $C_6 (> C_5)$ such that,

$$P(A_n) \le C_6 \frac{n}{n^{(1-\delta)(p+\varepsilon_1)+\delta} (\log n)^{p+\varepsilon_1}}$$
$$\sim \frac{C_6}{n(\log n)^{p+\varepsilon_1}} \tag{11}$$

and since $p+\varepsilon_1 > 1,\ P(A_n) \to 0$ as $n \to \infty$.

we have

$$\left\{A_n \cap A^c_{n+1}\right\} = \left\{M_{v_n} \ge C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}},\ M_{v_{n+1}} \le C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}\right\}$$

Observe that,

$$\left\{A_n \cap A^c_{n+1}\right\} = \left\{M_{v_n} \ge C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}},\ M_{v_n} \le C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}\right\}$$
$$\subset \left\{C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}} \le M_{v_n} \le C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}\right\}$$

which yields,

$$P\left(A_n \cap A^c_{n+1}\right) \le P\left(C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}} \le M_{v_n} \le C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}\right)$$
$$\le P\left(\frac{C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}}}{(v_n)^{\frac{1}{\alpha}}} \le \frac{M_{v_n}}{(v_n)^{\frac{1}{\alpha}}} \le \frac{C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}}{(v_n)^{\frac{1}{\alpha}}}\right)$$

Using the fact that $\frac{M_{v_n}}{(v_n)^{\frac{1}{\alpha}}} \overset{d}{=} X_1$ we have

$$P\left(A_n \cap A^c_{n+1}\right)$$
$$\leq P\left(\frac{C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}}}{(v_n)^{\frac{1}{\alpha}}} \leq X_1 \leq \frac{C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}}{(v_n)^{\frac{1}{\alpha}}} \right)$$
$$\leq P\left(y_n \leq X_1 \leq y_{n+1}\right) = \int_{y_n}^{y_{n+1}} f(x)dx,$$

Where $y_n = \frac{C_3 n^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p+\varepsilon_1}{\alpha}}}{(v_n)^{\frac{1}{\alpha}}}$ and

$$y_{n+1} = \frac{C_3 (n+1)^{\frac{(1-\delta)(p+\varepsilon_1)+\delta}{\alpha}} (\log(n+1))^{\frac{p+\varepsilon_1}{\alpha}}}{(v_n)^{\frac{1}{\alpha}}}.$$

Since f is the density function of a non-negative stable r.v. Density of stable law is given by, $f(x) = \frac{C_7}{x^{1+\alpha}} + \frac{C_8}{x^{1+2\alpha}} + o\left(\frac{1}{x^{1+2\alpha}}\right)$, where $C_7 > 0$ and $C_8 > 0$ are constants. Hence for x large, one can find $C > 0$ such that, $f(x) \leq C\left(\frac{1}{x^{1+\alpha}} + \frac{1}{x^{1+2\alpha}}\right)$.

Consequently for n large,

$$P\left(A_n \cap A^c_{n+1}\right) \leq C \int_{y_n}^{y_{n+1}} \left(\frac{1}{x^{1+\alpha}} + \frac{1}{x^{1+2\alpha}}\right) dx$$
$$\leq \frac{C}{\alpha}\left\{\left(\frac{1}{y_n^{\alpha}} - \frac{1}{y_{n+1}^{\alpha}}\right) + \frac{1}{2}\left(\frac{1}{y_n^{2\alpha}} - \frac{1}{y_{n+1}^{2\alpha}}\right)\right\}. \quad (12)$$

We have

$$y_{n+1}^{-\alpha} = \frac{C_9 v_n}{(n+1)^{(1-\delta)(p+\varepsilon_1)+\delta} (\log(n+1))^{p+\varepsilon_1}}$$

Similarly

$$y_n^{-\alpha} = \frac{C_{10} v_n}{n^{(1-\delta)(p+\varepsilon_1)+\delta} (\log n)^{p+\varepsilon_1}},$$

for some $C_9>0$ and $C_{10}>0$. Hence

$$y_n^{-\alpha} - y_{n+1}^{-\alpha} =$$
$$v_n \left[\frac{C_{10}}{n^{(1-\delta)(p+\varepsilon_1)+\delta} (\log n)^{p+\varepsilon_1}} - \frac{C_9}{(n+1)^{(1-\delta)(p+\varepsilon_1)+\delta} (\log(n+1))^{p+\varepsilon_1}} \right]$$
$$= \frac{v_n}{n^{(1-\delta)(p+\varepsilon_1)+\delta} (\log n)^{p+\varepsilon_1}} \left[C_9 - \frac{C_{10} n^{(1-\delta)(p+\varepsilon_1)+\delta} (\log n)^{p+\varepsilon_1}}{(n+1)^{(1-\delta)(p+\varepsilon_1)+\delta} (\log(n+1))^{p+\varepsilon_1}} \right]$$
$$\sim \frac{C_{11} n^{\delta}}{n^{(1-\delta)(p+\varepsilon_1)+\delta} (\log n)^{p+\varepsilon_1}}$$
$$\sim \frac{C_{11}}{n^{(1-\delta)(p+\varepsilon_1)} (\log n)^{p+\varepsilon_1}}.$$

Similarly we get

$$y_n^{-2\alpha} - y_{n+1}^{-2\alpha} \sim \frac{C_{12}}{n^{2(1-\delta)(p+\varepsilon_1)} (\log n)^{2(p+\varepsilon_1)}},$$

for some $C_{11}>0$ and $C_{12}>0$.

For n largesay $n \geq N$, from (12) we can find some constant $C_{13}(> C)$ such that,

$$P\left(A_n \cap A^c_{n+1}\right)$$
$$\leq \frac{C}{\alpha}\left\{\frac{C_{11}}{n^{(1-\delta)(p+\varepsilon_1)} (\log n)^{p+\varepsilon_1}} + \frac{1}{2}\left(\frac{C_{12}}{n^{2(1-\delta)(p+\varepsilon_1)} (\log n)^{2(p+\varepsilon_1)}}\right)\right\}$$
$$< \frac{C_{13}}{n^{(1-\delta)(p+\varepsilon_1)} (\log n)^{p+\varepsilon_1}}$$

Since $p + \varepsilon_1 > 1$, we have

$$\sum_{n\geq 3} P\left(A_n \cap A^c_{n+1}\right) < C_{13} \sum_{n\geq 3} \frac{1}{n^{(1-\delta)(p+\varepsilon_1)} (\log n)^{p+\varepsilon_1}} < \infty.$$

It follows that $\sum_{n\geq 3} P\left(A_n \cap A^c_{n+1}\right) < \infty$ and hence $P\left(A_n \cap A^c_{n+1} \text{ i.o.}\right) = 0$.

Which implies the proof of (7) and (5) by Lemma 2 and consequently proof of (1) follows from (5).

To prove (8), we need to prove that, for some d>0,

$$P\left(M_{u_n} \geq C_4 n^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n)^{\frac{p-\varepsilon_1}{\alpha}} \text{ i.o.}\right) \geq d > 0. \tag{13}$$

Using Lemma 1 (Extended Borel-Cantelli Lemma) and the Hewitt-Savage zero-one law, (2) will be proved.

Define $n_k = [k^\theta], 0 < \theta < 1$ and let

$$D_k = \left\{ C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \leq M_{u_{n_k}} \leq 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \right\}$$

Then,

$$P(D_k) = P\left(M_{u_{n_k}} \geq 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}}\right) - P\left(M_{u_{n_k}} \geq C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}}\right) \tag{14}$$

Consider,

$$P\left(M_{u_{n_k}} \geq 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}}\right) = P\left(M_{u_{n_k}} \geq y_{n_k}\right)$$

Where $y_{n_k} = 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}}$.

In view of (9), we can observe that, condition (2) of Heyde (1967) is satisfied by y_{n_k}. Again following proof of Lemma C in Vasudeva and Divanji (1991), we can show that, $\frac{M_{u_{n_k}}}{y_{n_k}} \xrightarrow{p} 0$ as $k \to \infty$.

Also since $1 \leq \limsup_{n\to\infty} \frac{y_{u_{n_k}}}{y_{n_k}} \leq (1+\varepsilon)^{\frac{1}{\alpha}}$, by Theorem in Hyede(1967), we can find some constants $C_{14}(>0)$ and k_1 such that, for all $k \geq k_1$,

$$P\left(M_{u_{n_k}} \geq y_{n_k}\right) \simeq C_{14} P\left(X_1 \geq y_{n_k}\right).$$

Again using (9), we can find some constant $C_{15}(>0)$ such that

$$P\left(M_{u_{n_k}} \geq y_{n_k}\right) \sim \frac{n_k C_{15}}{y_{n_k}^{\ \alpha}} = C_{15} \frac{n_k}{2n_k^{(1-\delta)(p-\varepsilon_1)+\delta} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}}} \sim \frac{C_{16}}{n_k^{-(1-\delta)(1+\varepsilon_1-p)} (\log n_k)^{p-\varepsilon_1}}$$

Therefore

$$P\left(M_{u_{n_k}} \geq y_{n_k}\right) \geq C_{16} \frac{n_k^{(1-\delta)(1+\varepsilon_1-p)}}{(\log n_k)^{p-\varepsilon_1}} \tag{15}$$

where C_{16} is some positive constant.

Similarly following above process, we can find some constant $C_{17}(>0)$ such that

$$P\left(M_{u_{n_k}} \geq \frac{y_{n_k}}{2}\right) \geq C_{17} \frac{n_k^{(1-\delta)(1+\varepsilon_1-p)}}{(\log n_k)^{p-\varepsilon_1}} \tag{16}$$

Substituting (15) and (16) in (14), we can find some constant $C_{18}(>0)$ and $k_2(>0)$ such that, for all $k \geq k_2$,

$$P(D_k) \geq C_{18} \frac{n_k^{(1-\delta)(1+\varepsilon_1-p)}}{(\log n_k)^{p-\varepsilon_1}},$$

where $n_k = [k^\theta], 0 < \theta < 1$. This implies that,

$$\sum_{k\geq k_2} P(D_k) \geq C_{18} \sum_{k\geq k_2} \frac{n_k^{(1-\delta)(1+\varepsilon_1-p)}}{(\log n_k)^{p-\varepsilon_1}} > C_{19} \sum_{k\geq k_2} \frac{k^{(1-\delta)(1+\varepsilon_1-p)}}{(\log k)^{p-\varepsilon_1}}, \tag{17}$$

For some constant $C_{19}(> C_{18})$ and hence, $\sum_{k\geq k_2} P(D_k) = \infty$.

Let $s > (\log k)^\eta, \eta > 1$, we have

$$\begin{aligned} P(D_k \cap D_s) &= P\Bigg(C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \leq M_{u_{n_k}} \leq 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}}, \\ &\qquad C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \leq M_{u_{n_s}} \leq 2C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \Bigg) \\ &\leq P(D_k)\, P\Bigg(M_{u_{n_s}} - M_{u_{n_k}} \geq C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} - 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \Bigg) \end{aligned}$$

Again following similar lines of (10) and Heyde'sTheorem , we have for $s > (\log k)^{\eta}, \eta > 1$,

$$P(D_k \cap D_s) \le P(D_k)\left(u_{n_s} - u_{n_k}\right).$$

$$P\left(X_1 \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}}\right)$$

Applying the arguments used to get the upper bound of $P(A_n)$, we can find some constants $C_{20}(>0)$ and $k_3(>0)$ whereby, for all $k \ge k_3$ and $s > (\log k)^{\eta}, \eta > 1$,

$$P(D_k \cap D_s) \le C_{20}\, P(D_k) P(D_s) \qquad (18)$$

Now for $(k+1) \le s \le (\log k)^{\eta}, \eta > 1$, we can note that,

$$\{D_k \cap D_s\} = \Big\{ C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \le M_{u_{n_k}} \le 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}},$$
$$C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \le M_{u_{n_s}} \le 2C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \Big\}$$
$$\subset \Big\{ C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \le M_{u_{n_k}} \le 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}},$$
$$M_{u_{n_s}} \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \Big\}$$

Which implies

$$P(D_k \cap D_s) \le P\Big(C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \le M_{u_{n_k}} \le 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}}$$
$$M_{u_{n_s}} \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \Big).$$

Observe that $M_{u_{n_k}}$ and $M_{u_{n_s}}$ are independent, we get,

$$P(D_k \cap D_s) \le P(D_k)\, P\left(M_{u_{n_s}} \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \right).$$

Following similar lines of (10), we can find some constants $C_{21}(>0)$ and $k_4(>0)$ such that, for all $k \ge k_4$,

$$P\left(M_{u_{n_s}} \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \right) \sim C_{21} n_s\, P\left(X_1 \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \right).$$

Using (9), we get some constants $C_{22}(> C_{21})$ and $k_5(>k_4)$ such that, for all $k \ge k_5$,

$$P\left(M_{u_{n_s}} \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \right) \sim \frac{C_{22}}{n_s^{-(1-\delta)\varepsilon_1} (\log n_s)^{p-\varepsilon_1}}$$

Using the fact that $s \ge k+1$, one can find some constants $C_{23}(>0)$ and $k_6(>0)$ such that, for all $k \ge k_6$,

$$P\left(M_{u_{n_s}} \ge C_4 n_s^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_s)^{\frac{p-\varepsilon_1}{\alpha}} \right) \le C_{23} \frac{k^{\theta\varepsilon_1(1-\delta)}}{(\log k)^{p-\varepsilon_1}}$$

Hence for all $k \ge k_6$,

$$P(D_k \cap D_s) \le C_{23} \frac{k^{\theta\varepsilon_1(1-\delta)}}{(\log k)^{p-\varepsilon_1}} P(D_k). \qquad (19)$$

Note that,

$$P(D_k) \le P\left(M_{u_{n_k}} \ge 2C_4 n_k^{\frac{(1-\delta)(p-\varepsilon_1)+\delta}{\alpha}} (\log n_k)^{\frac{p-\varepsilon_1}{\alpha}} \right)$$

Again following steps similar to the above process of (19), we can find some constants $C_{24}(>0)$ and $k_7(>0)$ such that, for all $k \ge k_7$, we have

$$P(D_k) \le C_{24} \frac{k^{\theta\varepsilon_1(1-\delta)}}{(\log k)^{p-\varepsilon_1}} \text{ for } s \ge k+1. \qquad (20)$$

From (19) and (20) there exists some constants $C_{25}(>0)$ and $k_8(>0)$ such that, for all $k \geq k_8$,

$$P\left(D_k \cap D_s\right) \leq C_{25} \frac{k^{2\theta\varepsilon_1(1-\delta)}}{(\log k)^{2(p-\varepsilon_1)}}.$$

Now

$$\sum_{k=1}^{n-1} \sum_{s=k+1}^{(\log k)^\eta} P\left(D_k \cap D_s\right) \leq C_{25} \sum_{k=1}^{n-1} \frac{(\log k)^\eta\, k^{2\theta_1(1-\delta)}}{(\log k)^{2(p-\varepsilon_1)}}$$

$$\leq C_{25} \sum_{k=1}^{n-1} \frac{k^{2\theta_1(1-\delta)}}{(\log k)^{2(p-\varepsilon_1)-\eta}}$$

This implies that for $n \geq N_1$, we have

$$\sum_{k=1}^{n-1} \sum_{s=k+1}^{(\log k)^\eta} P\left(D_k \cap D_s\right) \leq C_{25} (\log n)^{(1-2(p-\varepsilon_1)-\eta)} \quad (21)$$

From (17) we have for $n \geq N_2$,

$$\sum_{k=1}^{\infty} P(D_k) \geq C_{19} \frac{k^{(1-\delta)\varepsilon_1}}{(\log k)^{p-\varepsilon_1}} \geq C_{26} (\log n)^{\varepsilon_1} \quad (22)$$

for some $C_{26}>0$. Note that,

$$\frac{\sum_{k=1}^{n}\sum_{s=1}^{n} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} = \frac{2\sum_{k=1}^{n-1}\sum_{s=k+1}^{n} P\left(D_k \cap D_s\right) + \sum_{k=1}^{n} P(D_k)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} \quad (23)$$

By (22), the second term of the right side, tends to zero as $k \to \infty$. Hence consider,

$$\frac{2\sum_{k=1}^{n-1}\sum_{s=k+1}^{n} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2}$$

$$= \frac{2\sum_{k=1}^{n-1}\sum_{s=k+1}^{(\log k)^\eta} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} + \frac{2\sum_{k=1}^{n-1}\sum_{s=(\log k)^\eta+1}^{n} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} \quad (24)$$

By (21) and (22), we have,

$$\lim_{n\to\infty} \frac{2\sum_{k=1}^{n-1}\sum_{s=k+1}^{(\log k)^\eta} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} = 0. \quad (25)$$

And by (18) and (22), we get that,

$$\liminf_{n\to\infty} \frac{2\sum_{k=1}^{n-1}\sum_{s=(\log k)^\eta+1}^{n} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} \leq 2C_{20} > 0. \quad (26)$$

Using (25) and (26) in (23) we have,

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{s=1}^{n} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} \geq C_{27} > 0.$$

In view of (17) and (26), appealing to Lemma 1(Extended Borel-Cantelli Lemma) and Hewit-Savege zero-one law, we get $P(D_k i.o.)=1$. Hence the proof of the Theorem is completed.

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